MATB41 Week 1 Notes

1. Lines:

- Def: A line in R^n is decided by: - 2 points
  - A point and a direction

1. Lines in R^2:
   - Has the formula $ax+by=c$

   - 2-Point Eqn:
     Given 2 points, (x_1, y_1), (x_2, y_2), we can use the equation $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$
     to find the line that passes through them.

     We can rewrite the formula above to
     $y = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)+y_1$

     E.g. Find the eqn of the line that goes thru $(1,3)$ and $(8,0)$.

$$y = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)+y_1$$

$$= \left(\frac{0-3}{8-1}\right)(x-1)+3$$

$$= \left(\frac{-3}{7}\right)(x-1)+3$$

$$= -\frac{3}{7}x + \frac{3}{7} + 3$$

$$= -\frac{3}{7}x + \frac{24}{7}$$

$\therefore$ $y = -\frac{3}{7}x + \frac{24}{7}$ is the eqn of the line

- Vector Eqn:

A way to represent a line using a point and a direction.

Given a point P, we can find its position vector $\vec{P}$. Then, we can write

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\uparrow \ \vec{P}} + t \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\uparrow \ \vec{v}}$$

← Note: This is the vector eqn.

E.g. Find the eqn of a line that goes thru (1,1) and is parallel to the vector [2,5].

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

is the vector eqn of the line.

- Parametric Eqn:

Take the solution from above.

$$\begin{cases} x = 1 + 2t \\ y = 1 + 5t \end{cases} \quad t \in \mathbb{R}$$

← Parametric Eqn

Note: Vector and parametric eqns are not unique.

2. Lines in R^3:

- A line in R^3 is the intersection of 2 non-parallel planes.

- Vector and Parametric Eqns:
  - Vector and parametric eqns in R^3 are the same as in R^2.

  - E.g. Find a vector eqn and a parametric eqn of the line that passes thru the point $(1,1,0)$ and is parallel to the vector $v = [1,-3,-7]$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ -7 \end{bmatrix} \leftarrow \text{Vector eqn}$$

$$\begin{cases} x = 1+t \\ y = 1-3t \quad t \in R \leftarrow \text{Parametric eqn} \\ z = -7t \end{cases}$$

- Symmetric Eqn of a Line:
  - We know that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$x = x_0 + at \rightarrow t = \frac{x-x_0}{a}$$
$$y = y_0 + bt \rightarrow t = \frac{y-y_0}{b}$$
$$z = z_0 + ct \rightarrow t = \frac{z-z_0}{c}$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t \leftarrow \text{Symm Eqn}$$

Using the previous example, we see that

$$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-0}{-7} = t$$

E.g. Find a symm eqn for the line that passes thru $(1,1,0)$ and $(0,4,7)$. At what point does the line intersect the xz plane?

$$\vec{v_1} = (1,1,0) - (0,4,7)$$
$$= [1,-3,-7]$$

$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-0}{-7} = t$ is a symm eqn

$$\vec{v_2} = (0,4,7) - (1,1,0)$$
$$= [-1,3,7]$$

$\frac{x-0}{-1} = \frac{y-4}{3} = \frac{z-7}{7} = t$ is another symm eqn

Note! Symm eqns are not unique bc it doesn't matter what the order of the points are for $\vec{v}$ and it doesn't matter which point is used at the end.

The xz plane means $y=0$.
Using the first symm eqn,

$$\frac{y-1}{-3} = t \rightarrow y = -3t+1$$

$$0 = -3t+1 \rightarrow t = \frac{1}{3}$$

Subbing $t = \frac{1}{3}$ for x and z, we get

$$x - 1 = \frac{1}{3} \rightarrow x = \frac{4}{3}$$

$$\frac{z-0}{7} = \frac{1}{3} \rightarrow z = \frac{7}{3}$$

$(\frac{4}{3}, 0, \frac{7}{3})$ is the point of intersection.

2. Planes:
   - In R^3, $ax+by+cz=d$ is the eqn of a plane.
   - The intersection between 2 planes may be a line, if the planes are non-parallel, or may be empty, if the planes are parallel.
   - E.g. Find the intersection of
     $$\begin{cases} 2x+3y-z=4 \\ x-2y+z=1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 1 & -2 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 7 & -3 & 6 \end{array}\right]$$

Let $x_3 = t$, $t \in R$

$$y = \frac{6}{7} + \frac{3}{7}t$$
$$x = 2\left(\frac{6}{7} + \frac{3}{7}t\right) - t - 1$$
$$= \frac{5}{7} - \frac{1}{7}t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/7 \\ 6/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{7} \\ \frac{3}{7} \\ 1 \end{bmatrix}, \quad \begin{cases} x = \frac{5}{7} - \frac{1}{7}t \\ y = \frac{6}{7} + \frac{3}{7}t \\ z = t \end{cases} \quad t \in R$$